AN INVERSE THEOREM: WHEN $m(A+B) = m(A) + m(B)$ IN A LOCALLY COMPACT ABELIAN GROUP

JOHN T. GRIESMER

Abstract. M. Kneser described the scenario $m(A+B) < m(A) + m(B)$, when A and B are subsets of a locally compact abelian group G and m is Haar measure, showing in particular that $A+B$ is a union of cosets of a compact open subgroup. In the same article, Kneser classified the pairs (A, B) of sets satisfying $m(A+B) = m(A) + m(B)$, with the additional assumption that G is compact and connected. Recently, D. Grynkiewicz described the pairs of finite sets A, B satisfying $|A+B| = |A| + |B|$ in an abelian group. We combine the arguments of Grynkiewicz and Kneser to describe the pairs $A, B \subset G$ satisfying $m(A+B) = m(A) + m(B)$, in an arbitrary locally compact abelian group G.

1. Introduction

1.1. **Inverse theorems.** *Inverse theorems* in additive combinatorics deduce properties of subsets A, B of an abelian group from some hypothesis on the sumset $A+B := \{a+b : a \in A, b \in B\}$. The hypothesis on $A+B$ is often a bound on some quantity (perhaps cardinality, Haar measure, or some kind of density), in terms of the same quantities related to A and B. The books [11, 12] give a comprehensive introduction to a large segment of this theory. A very simple example of an inverse theorem says that if A and B are finite, and $|A+B| \leq \max\{|A|, |B|\}$, where $|S|$ is the cardinality of the set S, then one of A or B is a union of cosets of a subgroup $H \leqslant G$, and the other is contained in a coset of H. Kneser [8] described the pairs (A, B) of Haar measurable subsets of a locally compact abelian group G with Haar measure m satisfying $m_*(A+B) < m(A) + m(B)$. Here m_* denotes the inner measure corresponding to m, that is, $m_*(S) = \sup\{m(E) : E \subset S, E \text{ is compact}\}$. If $S \subset G$ and $t \in G$, then $S + t := \{s + t : s \in S\}$.

Theorem 1.1 ([8], Theorem 1)**.** *Let G be a locally compact abelian group with Haar measure m, and suppose $A, B \subset G$ are measurable and satisfy $m_*(A+B) < m(A) + m(B)$. Then the group $H := \{t : A+B+t = A+B\}$ is compact and open, $A+B = A+B+H$, and $m(A+B) = m(A+H) + m(B+H) - m(H)$.*

Since connected groups have no proper compact open subgroups, Theorem 1.1 implies $m_*(A+B) \geq \min\{m(A) + m(B), m(G)\}$ whenever G is connected.

In the same article, Kneser classified the subsets of a compact, *connected*, abelian group satisfying $m_*(A+B) = m(A)+m(B)$. We call such a pair (A, B) a *sur-critical pair*.

Theorem 1.2 ([8], Theorem 3)**.** *Let G be a compact connected abelian group, and suppose $A, B \subset G$ are measurable sets satisfying $m_*(A+B) = m(A) + m(B)$.*

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Then there is a continuous homomorphism $\chi : G \to \mathbb{R}/\mathbb{Z}$, *and there are intervals* $I, J \subset \mathbb{R}/\mathbb{Z}$ *such that* $A \subset \chi^{-1}(I), B \subset \chi^{-1}(J)$, *and* $m(A) = m\big(\chi^{-1}(I)\big)$, $m(B) = m\big(\chi^{-1}(J)\big)$.

1.2. **Summary of results.** In this article, we remove the hypothesis that G is connected from Theorem 1.2. The weaker hypothesis permits new types of sur-critical pairs, as shown in §1.6. Our main result is Theorem 1.4, which describes all sur-critical pairs for an arbitrary compact abelian group. Weakening the compactness hypothesis on G to local compactness produces no new examples, as shown in §4.

While the conclusion of Theorem 1.4 is trivial under the hypothesis that G is finite, the article [2] gives a very detailed description of sur-critical pairs for finite groups. The following discussion summarizes the development of some inverse theorems for finite groups and how these relate to Theorem 1.4.

Context. In contrast to inverse theorems, *direct theorems* deduce properties of $A+B$ from hypotheses on A and B. One of the earliest such results is the Cauchy-Davenport inequality, which states that $|A+B| \geq \min\{|A|+|B|-1, p\}$ whenever $A, B \subset \mathbb{Z}/p\mathbb{Z}$ for some prime p; here $|S|$ is the cardinality of the set S. The corresponding inverse theorem, due to Vosper, classifies the pairs (A, B) of subsets of $\mathbb{Z}/p\mathbb{Z}$ satisfying $|A+B| = |A|+|B|-1$, when $|A|+|B| < p$: the equation holds if and only if A and B are arithmetic progressions with the the same common difference, or one of $|A| = 1, |B| = 1$ [14, 13].

Generalizing Vosper's theorem (and strengthening the special case of Theorem 1.1 where G is discrete), Kemperman [7] described those pairs of finite subsets A, B of an abelian group which satisfy $|A+B| < |A|+|B|$. Recently, extensions of Vosper's and Kemperman's theorems for $|A+B| < |A|+|B|$, describing the case $|A+B| = |A|+|B|$, have appeared [3, 4]. More recently, Grynkiewicz [2] gave a very detailed description of those pairs (A, B) of finite subsets of an abelian group G satisfying $|A+B| = |A|+|B|$. The description given in [2] is somewhat intricate, so we do not reproduce it here. In Theorem 1.4, we describe those pairs (A, B) of subsets of a locally compact abelian group G satisfying $m(A+B) = m(A)+m(B)$. Concatenating Theorem 1.4 with the results of [7] and [2] yields a very precise description of pairs (A, B) of subsets of a locally compact abelian group satisfying $m_*(A+B) = m(A)+m(B)$, but we will not state this description explicitly.

Theorem 1.4 was developed partly to answer Question 4.1 of [6].

Background. We assume knowledge of the theory of locally compact abelian groups, in particular the fact that Haar measure can be disintegrated over the cosets of a compact subgroup; a compact abelian group G has either a neighborhood base at the identity consisting of subgroups, or a surjective character $\chi : G \to \mathbb{R}/\mathbb{Z}$; and subgroups of G having positive Haar measure are open. The references [9], [5], and [1] each provide sufficient background, as does [8].

1.3. **Terminology and notation.** Throughout, G will denote a locally compact abelian group and m_G will be its Haar measure. When there is no chance of confusion, we write m for m_G. The term *measurable* in reference to a subset of G will always mean "lies in the completion of the Borel σ-algebra with respect to m." Haar measure will always be normalized for compact groups, so $m(G) = 1$ for such

G. The symbol $\mathbb{T}$ will denote the group $\mathbb{R}/\mathbb{Z}$, and "an interval in $\mathbb{T}$" means a set of the form $[x,y]+\mathbb{Z}$, where $x \leq y < x+1$.

If $S \subset G$, S^c will denote the complement $G \setminus S$.

If $m_*(A+B) = m(A) + m(B)$, then (A,B) is called a *sur-critical pair*. We call a pair (A,B) satisfying $m_*(A+B) < m(A) + m(B)$ a *critical pair*.

If $C, D \subset G$, write $C \sim D$ if $m(C \triangle D) = 0$. When C is measurable, the group $H(C) := \{t \in G : (C+t) \sim C\}$ is a compact subgroup of G ([8], Lemma 4).

Write $C \subset_m D$ or $D \supset_m C$ if $m(D \setminus C) = 0$.

If $H \leqslant G$ is a compact open subgroup, $\phi_H : G \to G/H$ will denote the quotient map. We may identify subsets of G of the form $A + H$ with subsets of G/H.

An *H-coset decomposition* of a set $A \subset G$ is the collection of sets $A \cap H_i$, where H_i ranges over the cosets of H that meet A.

We say $c \in A + B$ is a *unique expression element* if $c = a_0 + b_0$ for some $a_0 \in A, b_0 \in B$, and $a + b = c$ implies $a = a_0, b = b_0$ for $a \in A, b \in B$.

Much of our terminology is taken from [2]; when G is discrete, some of our definitions coincide with those from [2].

Periodicity. If $A \sim A + H$ for some some compact open subgroup $H \leqslant G$, we call A *periodic* with period H. Otherwise, we call A *aperiodic*. Note that the assertion $H(A) = \{0\}$ implies A is aperiodic, but an aperiodic A may have $m\big(H(A)\big) > 0$.

Extendibility and reducibility. Let A and B be measurable subsets of a compact abelian group G. We say that A is *extendible with respect to B* if there is a measurable set $A' \supset A$ with $m(A') > m(A)$ and $m_*(A' + B) = m_*(A + B)$. We say that the pair (A,B) is *extendible* if A is extendible with respect to B or B is extendible with respect to A. Otherwise, we say that (A,B) is *nonextendible*.

If there are subsets $A' \subset A$ and $B' \subset B$ such that $m(A') = m(A)$ and $m(B') = m(B)$, while $m_*(A' + B') < m(A') + m(B')$, we say that (A,B) is *reducible*. If $m_*(A' + B') = m_*(A + B)$ for all $A' \subset A, B' \subset B$ having $m(A') = m(A)$ and $m(B') = m(B)$, then (A,B) is called *irreducible*.

Quasi-periodicity. Let $H \leqslant G$ be a compact open subgroup. A set $A \subset G$ is called *quasi-periodic with respect to H* if A can be partitioned into two sets $A = A_1 \cup A_0$ such that $(A_1 + H) \cap (A_0 + H) = \emptyset$, $A_1 \sim A_1 + H$, and A_0 is contained in a coset of H. The group H is called a *quasi-period* of A. Note that A may have more than one quasi-period.

We say that a pair (A,B) of subsets of G has a *quasi-periodic decomposition with respect to H* if one (or both) of A or B is quasi-periodic with respect to H and the other is either contained in a coset of H or is quasi-periodic with respect to H.

Complementary pairs. If G is compact and $m(A+B) = m(A) + m(B) = 1$, call (A, B) a *complementary* pair. When G is infinite, it is easy to construct such pairs (A, B) with $A + B \neq G$: let $A \subset G$ be any set meeting every coset of every finite index subgroup of G, with $0 < m(A) < 1$. Then $(A, -A^c)$ is a complementary pair by Theorem 1.1.

Complementary pairs (A, B) satisfying $A + B \neq G$ can be described as follows. If (A, B) is a complementary pair, and $A + B \neq G$, then $A \cap (t - B) = \emptyset$ for some $t \in G$, so $t - B \sim A^c$. If $s \notin H(t - B)$ $(= H(A^c) = H(A))$, then $m(A \cap (s - B)) > 0$, so $s \in A + B$. It follows that $A + B$ contains all of $G \setminus (t + H(A))$.

If $H \leqslant G$ is a compact open subgroup, A and B are each contained in a coset of H, and $m(A + B) = m(A) + m(B) = m(H)$, we say that (A, B) is *complementary with respect to* H.

1.4. **Topology of** $A, B,\ A+B$, **and their closures.** Let $\overline{S}$ denote the topological closure of the set S, and let $\operatorname{int} S$ denote the interior of S.

Theorem 1.2 yields topological information about A and B when G is connected: if (A, B) is a sur-critical pair for a compact connected group, then

$$m(\overline{A}) = m(\operatorname{int} \overline{A}) = m(A), \tag{1.1}$$

$$m(\overline{B}) = m(\operatorname{int} \overline{B}) = m(B), \text{ and} \tag{1.2}$$

$$m(\overline{A+B}) = m(\operatorname{int} \overline{A+B}) = m(A+B). \tag{1.3}$$

Call a set A *essentially regular* if it satisfies (1.1), and call a pair (A, B) *essentially regular* if A and B satisfy (1.1)-(1.3).

Lemma 1.3. *If for all* $\varepsilon > 0$, (A, B) *has a quasi-periodic decomposition with respect to a compact open subgroup* H *with* $0 < m(H) < \varepsilon$, *then* (A, B) *is essentially regular.*

Proof. For a common quasi-period H with $0 < m(H) < \varepsilon$, we have

$$m(\overline{A}) \leq m(A + H) \leq m(A) + \varepsilon,$$

and

$$m(\operatorname{int} \overline{A}) \geq m(A + H) - m(H) \geq m(A) - \varepsilon.$$

Letting $\varepsilon \to 0$, we get (1.1). The same argument applied to B and $A + B$ implies (1.2) and (1.3). □

Theorem 1.4 will show that most sur-critical pairs are essentially regular, and will give a simple description of those sur-critical pairs which are not essentially regular.

1.5. **Description of sur-critical pairs.** Our main result, Theorem 1.4, describes the sur-critical pairs for a compact abelian group. Corollary 4.4 will show how sur-critical pairs for a locally compact abelian group G may be obtained from sur-critical pairs in a compact quotient of G.

Theorem 1.4. *Let* G *be a compact abelian group, and let* $A, B \subset G$ *be measurable sets with* $m(A), m(B) > 0$, *satisfying* $m_*(A + B) = m(A) + m(B)$. *Then at least one of the following is true*:

(P) *There is a compact open subgroup* $H \leq G$ *with* $A + H \sim A$ *and* $B + H \sim B$.

(E) *There are measurable sets $A' \supset A$ and $B' \supset B$ such that*

$$m(A') + m(B') > m(A) + m(B)$$

and $m(A' + B') = m(A + B)$.

(K) *There is a compact open subgroup $H \leqslant G$, a surjective continuous homomorphism $\chi : H \to \mathbb{T}$, intervals $I, J \subset \mathbb{T}$, and $a, b \in G$ such that $A \subset a + \chi^{-1}(I)$, $B \subset b + \chi^{-1}(J)$, $m(A) = m(\chi^{-1}(I))$, and $m(B) = m(\chi^{-1}(J))$.*

(QP) *There is a compact open subgroup $H \leqslant G$ such that $A = A_1 \cup A_0$ and $B = B_1 \cup B_0$, where $A_1 + H \sim A_1$, $B_1 + H \sim B_1$, while A_0, B_0 are each contained in a coset of H, $(A_1 + H) \cap (A_0 + H) = (B_1 + H) \cap (B_0 + H) = \emptyset$, and $A_0 + B_0 + H$ is a unique expression element of $A + B + H$ in G/H. Furthermore $m_*(A_0 + B_0) = m(A_0) + m(B_0)$.*

In conclusions (P), (E), *and* (K), *$A + B$ is measurable. In conclusion* (QP), *$A + B$ may fail to be measurable, and then there exists a compact open subgroup $H \leq G$, and partitions $A = A_1 \cup A_0$, $B = B_1 \cup B_0$ satisfying* (QP) *where one of $m(A_0) = 0$ or $m(B_0) = 0$.*

The labels (P), (E), (K), and (QP) stand for "periodic", "extendible," "Kneser," and "quasi-periodic," respectively.

Theorem 1.4 is proved in §3.2. A sequence of lemmas in §2 reduces the proof to the case where (A, B) is an irreducible, nonextendible, sur-critical pair with $A + B$ measurable and $H(A + B) = \{0\}$. We then apply the e-transform as in the proof of Theorem 1.2 from [8]. Since we do not assume G is connected, new complications may arise in applying the e-transform. The most difficult of these is handled by Lemma 2.18, whose proof is modeled after a similar proof in [2]. Lemma 3.1 gives a rough outline of our proof of Theorem 1.4.

Remark 1. If (A, B) is a sur-critical pair satisfying (E), then one of the following holds.

(E.1) $A' + B' = A + B$.

(E.2) A, B, and $A + B$ each have quasi-periodic decompositions with respect to $H := H(A + B)$. These decompositions $A = A_1 \cup A_0, B = B_1 \cup B_0$, satisfy $A_1 + B_1 + H = A_1 + B_1$, and $A_0 + B_0 + H$ is a unique expression element of $A + B + H$ in G/H. Possibly $A_1 = B_1 = \emptyset$, in which case (A, B) is complementary with respect to the subgroup H.

This classification is a fairly straightforward application of Theorem 1.4, and we omit the proof.

Remark 2. Sur-critical pairs satisfying (QP) can be further described in terms of critical pairs. We first describe a process for producing sur-critical pairs satisfying (QP) from critical pairs, then we show that every sur-critical pair satisfying (QP) arises from this process.

Let G be a compact abelian group with finite quotient $F = G/H$. Let (A', B') be a critical pair for F such that $|A' + B'| = |A'| + |B'| - 1$, and let $a' \in A', b' \in B'$ such that $a' + b'$ is a unique expression element of $A' + B'$. Let $A_1' = A' \setminus \{a'\}, B_1' = B' \setminus \{b'\}$. Set $A_1 = A_1' + H$, $B_1 = B_1' + H$, and choose $C \subset H$ and $D \subset H$ such that

$m_*(C+D) = m(C) + m(D)$. Let $A_0 = a' + C, B_0 = b' + D$, and let $A = A_1 \cup A_0$, $B = B_1 \cup B_0$. We then have

$$m(A) = m(H)(|A'| - 1) + m(C),$$
$$m(B) = m(H)(|B'| - 1) + m(D),$$

and $m_*(A+B) = m(H)(|A'+B'| - 1) + m_*(C+D) = m(A) + m(B)$.

To see that every sur-critical pair satisfying (QP) arises this way, let $A = A_1 \cup A_0$, $B = B_1 \cup B_0$ constitute such a pair, and let H be corresponding quasi-period. Set $A' = A+H, B' = B+H$, so that $m(A') = m(A_1)+m(H)$, $m(B') = m(B_1)+m(H)$, and

$$\begin{aligned} m(A'+B') &= m\big((A_1+B)+(A+B_1)\big) + m(H) \\ &= m(A+B) - m(A_0+B_0) + m(H) \\ &= m(A) - m(A_0) + m(B) - m(B_0) + m(H). \end{aligned}$$

Viewing A' and B' as subsets of G/H, we have $|A'+B'| = |A'| + |B'| - 1$, while A and B may be constructed from A' and B' as in the previous paragraph.

Remark 3. While infinite disconnected groups G admit sur-critical pairs (A, B) such that $A+B$ is not measurable, Theorem 1.4 implies that there are always $A' \subset A, B' \subset B$ which are countable unions of compact sets such that $m(A'+B') = m_*(A+B)$.

Notice that in conclusion (QP), we may have $m(A_0) = 0, m(B_0) = 0$, or both. When $m(A_0) > 0$ and $m(B_0) > 0$, we can apply Theorem 1.4 to the pair (A_0, B_0), and repeat if possible. With this observation, we can guarantee essential regularity (see §1.4) of sur-critical pairs by imposing conditions on the measures $m(A)$ and $m(B)$.

Corollary 1.5. *If (A, B) is a sur-critical pair for a compact abelian group G such that $m(A), m(B)$, and $m(A) + m(B)$ are all irrational numbers, then (A, B) is essentially regular.*

Proof. With these hypotheses, an application of Theorem 1.1 shows that only conclusions (K) and (QP) can hold in Theorem 1.4. If (K) holds, then it is easy to check that (A, B) is essentially regular. Assume (QP) holds. Then $m(A_1), m(B_1)$ are rational, while $m(A_0), m(B_0)$, and $m_*(A_0+B_0)$ are irrational. By induction, either (A, B) has quasi-periodic decompositions with respect to compact open subgroups H of arbitrarily small measure, or (A, B) has a quasi-periodic decomposition $A = A_1 \cup B_0$, $B = B_1 \cup B_0$, where (A_0, B_0) satisfies (K). In the first case, Lemma 1.3 then implies (A, B) is essentially regular. In the second, the essential regularity follows from the fact that (A_0, B_0) is essentially regular, while $A_1 \sim A_1 + H$, $B_1 \sim B_1 + H$. □

1.6. **Examples.** We list some examples of sur-critical pairs to show that each alternative in Theorem 1.4 can occur, and that none of the alternatives can be omitted from the description.

Examples arising from finite groups. If F is a finite group, every sur-critical pair for F satisfies (P), and every nonextendible sur-critical pair satisfies (P) but not (E) or (K). Every periodic sur-critical pair (A, B) for a group G has the form $(\phi^{-1}(C), \phi^{-1}(D))$, where (C, D) is a sur-critical pair for a finite group F, and $\phi : G \to F$ is a homomorphism, so the periodic sur-critical pairs for an arbitrary group G are described in [2].

Complementary pairs and extendible pairs. Examples satisfying (E) may be constructed in a straightforward way from critical pairs in infinite compact abelian groups. Specifically, let G be an infinite compact abelian group with a proper compact open subgroup H. Let $C, D \subset G/H$ satisfy $|C + D| = |C| + |D| - 1$, and let $c_0 + d_0$ be a unique expression element of $C + D$. Write $C_1 = C \setminus \{c_0\}$, $D_1 = D \setminus \{d_0\}$. Let $A_0' \subset H$, set $B_0' = -(H \setminus A_0')$, and further insist that A_0' has nonempty intersection with every coset of every compact open subgroup $K \leq H$. Let $A_1 = C_1 + H$, $B_1 = D_1 + H$, and let $A_0 = c_0 + A_0'$, $B_0 = d_0 + B_0'$. By Theorem 1.1, $m(A_0' + B_0') = m(A_0') + m(B_0')$. Then $m(A + B) = m(H)(|C| + |D| - 1) + m(A_0) + m(B_0) = m(H)(|C| + |D|) = m(A) + m(B)$. Generically, a pair (A, B) so constructed will not satisfy (P) or (K), but will satisfy (E.2).

To form a specific example of the above construction, let $G = (\mathbb{Z}/15\mathbb{Z}) \times \mathbb{T}$, let $A = \big(\{1, 3, 5\} \times \mathbb{T}\big) \cup \big(\{7\} \times S\big)$, where $S \subset \mathbb{T}$ is any measurable set. Let $B = \big(\{0, 2\} \times \mathbb{T}\big) \cup \big(\{4\} \times (-S^c)\big)$.

To find an example satisfying (E.1), but not (E.2), (P), (K), or (QP), let $A'', B'' \subset G$ satisfy $m(A''+B'') = m(A''+H)+m(B''+H)-m(H)$, where $H = H(A''+B'')$, with $m(H) > 0$. Choose any pair of subsets $A \subset A'', B \subset B''$ with $m(A)+m(B) = m(A'') + m(B'') - m(H)$, such that A meets every coset of every finite index subgroup contained in $A + H$, B meets every coset of every finite index subgroup contained in $B + H$, and one of A or B does not have a quasi-periodic decomposition. Then (A, B) is the promised example. A specific example of this construction with $G = (\mathbb{Z}/17\mathbb{Z}) \times \mathbb{T}$ is given by $A = \{1, 3, 5, 7\} \times [0, 0.8]$, $B = \{0, 2\} \times [0, 0.9]$.

Pairs arising from $\mathbb{T}$. Let G be a compact group which is not totally disconnected. Let (A, B) have the form $A = \chi^{-1}(I), B = \chi^{-1}(J)$, where $H \leqslant G$ is a compact open subgroup, $\chi : H \to \mathbb{T}$ is a continuous surjective homomorphism, $I, J \subset \mathbb{T}$ are intervals, and $m_{\mathbb{T}}(A) + m_{\mathbb{T}}(B) < 1$. Clearly (A, B) is a sur-critical pair satisfying (K) but not (P), (E), or (QP).

A quasi-periodic pair. This example will satisfy (QP) but not (P), (E), or (K).

Let $G = \mathbb{Z}_7$, the 7-adic integers with the usual topology. Define the set $C \subset \mathbb{Z}$ by

$$C := (\{0, 1\} + 7\mathbb{Z}) \cup \Big(2 + 7\big((\{0, 1\} + 7\mathbb{Z}) \cup (2 + 7(\cdots))\big)\Big),$$

so that $C = (\{0, 1\} + 7\mathbb{Z}) \cup (2 + 7C)$. Let A be the closure of C in $\mathbb{Z}_7$, and let $B = A$. Then $m(A) = m(B) = \sum_{n=1}^{\infty} \frac{2}{7^n} = 1/3$. Note that A has a quasi-periodic decomposition $A_1 = \overline{\{0, 1\} + 7\mathbb{Z}}, A_0 = A \setminus A_1$. The sumset $A + B$ is the closure of $C + C$ in $\mathbb{Z}_7$. Note that

$$C + C = (\{0, 1, 2, 3\} + 7\mathbb{Z}) \cup \Big(4 + 7\big((\{0, 1, 2, 3\} + 7\mathbb{Z}) \cup (4 + 7(\cdots))\big)\Big),$$

so $m(A+B) = 2/3 = m(A)+m(B)$. The description of $A+B$ shows that $H(A+B) = \{0\}$, so (P) and (E) fail. Since G is totally disconnected, (K) cannot hold.

An irregular pair. Let G_0 be an infinite compact abelian group with Haar measure m_0, and let $G = (\mathbb{Z}/4\mathbb{Z}) \times G_0$. Let $A_0', B_0' \subset G_0$ with $m_{0*}(A_0' + B_0') = m_0(A_0') + m_0(B_0')$, let $A_1 = \{0\} \times G_0, B_1 = \{0\} \times G_0$, and let $A_0 = \{1\} \times A_0', B_0 = \{1\} \times B_0'$. Let $A = A_1 \cup A_0, B = B_1 \cup B_0$. Then $m_*(A+B) = m(A) + m(B)$, but depending on our choice of A_0 and B_0, $A+B$ may not be measurable. If A_0' is a singleton, then B_0' may be an arbitrary measurable subset of G_0.

2. Lemmas

We fix a compact abelian group G with Haar measure m. Unless stated otherwise, sets A and B are assumed to be subsets of G.

2.1. **Reduction to a special case.** Lemmas 2.1–2.11 will reduce the proof of Theorem 1.4 to the case where the sur-critical pair (A, B) is irreducible and nonextendible (see §1.3 for terminology), the sumset $A+B$ is measurable, and $H(A+B) = \{0\}$.

Lemmas 2.3 and 2.4 and Corollary 2.5 deal with the case where (A, B) is reducible, that is, where there exist $A' \subset A$ and $B' \subset B$ such that $m(A') + m(B') = m(A) + m(B)$ and $m(A' + B') < m(A') + m(B')$. Lemmas 2.6 and 2.7 handle technicalities that arise repeatedly in later arguments. Lemma 2.8 reduces the proof of Theorem 1.4 to the case where $A + B$ is measurable, and Lemma 2.11 further reduces the proof to the case where $H(A + B) = \{0\}$.

Lemma 2.1. *If $m(A), m(B) > 0$ and $m(A + B) < m(A) + m(B) \leq 1$, then for all $a \in A, b \in B$,*

$$m\big((a + H) \cap A\big) + m(A + B) \geq m(A) + m(B),$$
$$m\big((b + H) \cap B\big) + m(A + B) \geq m(A) + m(B),$$

where $H = H(A + B)$.

Proof. Write $m(A) = \sum_{i=1}^{n} m(A_i)$, where $A = \bigcup_{i=1}^{n} A_i$ is an H-coset decomposition of A. Then for each j, $m(A_j) = m(A) - \sum_{i\neq j} m(A_i)$, so Theorem 1.1 implies

$$\begin{aligned} m(A_j) + m(A + B) &= \Big(m(A) - \sum_{i\neq j} m(A_i)\Big) + m(A + H) + m(B + H) - m(H) \\ &= \Big(m(A) - \sum_{i\neq j} m(A_i)\Big) + \Big(\sum_{i} m(H)\Big) + m(B + H) - m(H) \\ &= m(A) + \Big(\sum_{i\neq j} m(H) - m(A_i)\Big) + m(H) + m(B + H) - m(H) \\ &\geq m(A) + m(B + H) \\ &\geq m(A) + m(B). \end{aligned}$$

The second assertion follows by reversing the roles of A and B. □

We will need a corollary of Theorem 1.1. We write $C \subset_m D$ or $D \supset_m C$ when $m(C \setminus D) = 0$.

Corollary 2.2. *Suppose* $m(A+B) < m(A) + m(B) \leq 1$*, and* $H := H(A+B)$*. Then*

$$\{z \in G : z + B \subset_m A + B\} = \{z \in G : z + B \subset A + B\} = A + H,$$

and

$$\{z \in G : z + A \subset_m A + B\} = \{z \in G : z + A \subset A + B\} = B + H.$$

Proof. For the first equality, note that $z + B \subset A + B$ is equivalent to $z + B \subset_m A+B$, by Lemma 2.1. For the second equality, note that if $z \in G$ with $z+B \subset A+B$, but $z \notin A+H$, then $z+H+B \subset A+B$. Then $A'+B = A+B$, where $A' = A\cup(z+H)$, and $m(A' + H) = m(A + H) + m(H)$. By Theorem 1.1, applied to (A', B),

$$\begin{aligned} m(A' + B) &= m(A' + H) + m(B + H) - m(H) \\ &\geq m(A + H) + m(B + H), \end{aligned}$$

which contradicts the assumption that $m(A + B) < m(A) + m(B)$. □

Lemma 2.3. *If* (A, B) *is a sur-critical pair and there exists* $A' \subset A$ *such that* $m(A') = m(A)$ *and* $m(A' + B) < m(A + B)$*, then* $A' + H(A' + B) \subset_m A$ *and at least one of the following holds.*

(i) (A, B) *has a quasi-periodic decomposition with respect to* $H(A' + B)$*.*

(ii) $B \sim B + H(A' + B)$*.*

If (A, B) *is nonextendible, then* (i) *holds.*

Proof. If A, B, and A' are as in the hypothesis, Theorem 1.1 says that $H := H(A'+B)$ is compact and open, and $m(A'+B) = m(A'+H)+m(B+H)-m(H)$; in particular $A' + B$ is a union of cosets of H. Lemma 2.1 then implies

$$m\big(B \cap (b + H)\big) + m(A' + B) \geq m(A) + m(B) \text{ for all } b \in B. \tag{2.1}$$

In particular, $m\big(B \cap (b + H)\big) > 0$ for all $b \in B$. Thus, if $a \in A$ is such that $a + B \not\subset A' + B$, then

$$m\big((a + B) \setminus (A' + B)\big) > 0. \tag{2.2}$$

Let $B_0 = (b_0 + H) \cap B$ be such that $A + B_0 \not\subset A' + B$. Let $B_1 = B \setminus B_0$.

Claim 1. $A' + H \subset_m A$ and $B_1 + H \subset_m B$.

Proof of Claim 1. By (2.1) and the definition of B_0 we can write the measure of $A + B$ as $m(A + B) = m(A' + B) + m(B_0)$. Then

$$\begin{aligned} m(A') + m(B) &= m(A + B) \\ &= m(A' + B) + m(B_0) \\ &= m(A' + H) + m(B + H) - m(H) + m(B_0) \\ &= m(A' + H) + m(B_1 + H) + m(B_0). \end{aligned}$$

Collecting terms, we find

$$[m(A' + H) - m(A')] + [m(B_1 + H) + m(B_0) - m(B)] = 0$$

Since each bracketed summand above is nonnegative, we have

$$m(A') = m(A' + H) \tag{2.3}$$

and

$$m(B) = m(B_1 + H) + m(B_0). \tag{2.4}$$

Since $B \subset (B_1 + H) \cup B_0$, (2.3) and (2.4) prove the claim. □

We now proceed based on whether $m(B_0) < m(H)$.

If $m(B_0) = m(H)$, we have $B \sim B + H$, and we conclude (ii).

Now assume $m(B_0) < m(H)$, and let $A_0 := A \setminus (A' + H)$. Fix $a_0 \in A$ such that $a_0 + B_0 \not\subset A' + B$.

Claim 2. A_0 is contained in a coset of H.

Proof of Claim 2. If $a, a' \in A$ are such that $a+B_0 \not\subset A'+B$ and $a'+B_0 \not\subset A'+B$, then $a+B_0 \sim a'+B_0$, by (2.1). Since B_0 is contained in a coset of H, this similarity implies $a - a' \in H$. We conclude that A_0 is contained in a coset of H. □

Writing $A_1 = A'+H$, and maintaining A_0, B_1, and B_0 as defined above, we have quasi-periodic decompositions $A = A_1 \cup A_0$ and $B = B_1 \cup B_0$.

If (A, B) is nonextendible, then $m(B_0) < m(H)$ or $B + H \sim B$. In the first case, we have the quasi-periodic decomposition of (A, B) by Claim 2, and in the second we have $A \sim A + H$ and $B \sim B + H$. The second alternative contradicts $m(A' + B) < m(A + B)$, so we conclude (i). □

Lemma 2.4. *If (A, B) is a reducible nonextendible sur-critical pair, then (A, B) has a quasi-periodic decomposition.*

Proof. Let $A' \subset A$ and $B' \subset B$ such that $m(A') + m(B') = m(A) + m(B)$ and $m(A' + B') < m(A + B)$. Write $H := H(A' + B')$.

If $m(A' + B) < m(A + B)$ or $m(A + B') < m(A + B)$, we may apply Lemma 2.3 to the pair (A, B) to obtain the conclusion. Otherwise, we have $m(A' + B) = m(A+B') = m(A+B)$. If both (A, B') and (A', B) are nonextendible, then we apply Lemma 2.3 to the pairs (A, B') and (A', B) to obtain quasi-periodic decompositions of A and B with respect to $H(A' + B')$.

Now suppose $m(A' + B) = m(A + B') = m(A + B)$, and one of the pairs (A, B') or (A', B) is extendible. If (A, B') is extendible, then by Lemma 2.3

$$B' \sim B' + H$$

and we may assume that $A' \sim A' + H$ (by passing to a subset of A', if necessary). We claim that $A_0 := A \setminus (A' + H)$ is contained in a coset of H. Let t be the number of cosets of H occupied by $A \setminus (A' + H)$, so that $m(A + H) - m(A') = t \cdot m(H)$. We consider the pair $(A + H, B' + H)$. Since $B' + H \sim B$, we have $A + H + B' + H \sim A + B' \sim A + B$. Assuming $t > 0$, we have

$$\begin{aligned} m\big((A + H) + B'\big) &= m(A + B) \\ &= m(A' + H) + m(B') \\ &= m(A + H) - t \cdot m(H) + m(B') \\ &< m(A + H) + m(B'). \end{aligned} \tag{2.5}$$

Applying Theorem 1.1 to the pair $(A+H, B')$ yields a compact open subgroup $K \leq G$ such that

$$(A+H)+B'+K \sim (A+H)+B' \tag{2.6}$$

and

$$m(A+H+B') = m(A+H+K) + m(B'+K) - m(K). \tag{2.7}$$

The nonextendibility of (A, B), (2.5), and (2.6) together imply $B'+K \subset_m B'$. Then $A'+B' \sim A'+B'+K$, so $K \leq H$ by the definition of H. Now (2.7) becomes

$$m(A+H+B') = m(A+H) + m(B') - m(K). \tag{2.8}$$

Comparing (2.8) with the third line of (2.5), we get $t \cdot m(H) = m(K)$. This implies $t = 0$ or $t = 1$, since $K \leq H$.

We have shown that A_0 is contained in a coset of H, so $A_1 := A \setminus A_0$ gives the desired quasi-periodic decomposition of A. To get a quasi-periodic decomposition of B, we proceed based on whether (A', B) is extendible. If (A', B) is extendible, we apply the preceding argument with B in place of A. If (A', B) is nonextendible, we apply Lemma 2.3 to the pair (B, A') and obtain the conclusion. □

Corollary 2.5. *If (A, B) is a reducible sur-critical pair and $A+B$ is aperiodic, then (A, B) has a quasi-periodic decomposition satisfying conclusion* (QP) *of Theorem* 1.4.

Proof. By Lemma 2.4 (A, B) has a quasi-periodic decomposition $A = A_1 \cup A_0$, $B = B_1 \cup B_0$. We need only show that $A_0 + B_0$ is a unique expression element of $A+B+H$ in G/H. Assuming otherwise, we find that $A+B$ is periodic, contradicting our assumption. □

Lemma 2.6. *Suppose (A, B) is an irreducible nonextendible sur-critical pair, and $H := H(\tilde{A}+\tilde{B})$ has positive Haar measure for some $\tilde{A} \subset A, \tilde{B} \subset B$ with $m(\tilde{A}) = m(A)$ and $m(\tilde{B}) = m(B)$. Then either A and B are periodic with period H, or (A, B) has a quasi-periodic decomposition with respect to H satisfying conclusion* (QP) *of Theorem* 1.4.

Proof. Let $\delta = m(H)$. Since $m(H(\tilde{A}+\tilde{B})) > 0$, we have $m_*((A+B) \cap H_i) \in \{0, \delta\}$ for every coset H_i of H. This fact, combined with the nonextendibility of (A, B), implies $m(A \cap H_i) \in \{0, \delta\}$ and $m(B \cap H_i) \in \{0, \delta\}$ for every coset H_i of H. Now by the irreducibility of (A, B), there are sets $A' \subset A, B' \subset B$ such that $A \sim A'+H$, $B \sim B'+H$, $A'+B'$ is measurable, and $m_*(A+B) = m(A'+B')$. We claim that $A_0 := A \setminus (A'+H)$ is contained in a coset of H. To prove this claim, assume A_0 is nonempty, and let $a \in A_0$. Then $a+B'+H \subset_m A+B$, so $[A' \cup (a+H)]+B' \subset_m A+B$. This implies

$$m\big([A' \cup (a+H)]+B'\big) = m(A+B) < m(A' \cup (a+H)) + m(B').$$

Corollary 2.2, applied to the pair $(A' \cup (a+H), B')$, implies

$$(A'+H) \cup (a+H) = \{z : z+B' \subset [A' \cup (a+H)]+B'\}.$$

The last set is just $\{z : z+B' \subset_m A'+B'\}$, which does not depend on $a \in A_0$. Hence, $(A'+H) \cup (a+H)$ does not depend on the choice of $a \in A_0$. This means that A_0 is contained in a coset of H, as claimed.

We set $A_1 = A \setminus A_0$ to get the desired quasi-periodic decomposition $A_1 \cup A_0$ of A with $m(A_0) = 0$. Reversing the roles of A and B, we find that either B is H-periodic or has an H-quasi-periodic decomposition $B_1 \cup B_0$ with $m(B_0) = 0$. If one of A_0 or B_0 is nonempty, the nonextendibility of (A, B) now implies that both A_0 and B_0 are nonempty, and $A_0 + B_0 + H$ is a unique expression element of $A + B + H$ in G/H. □

Lemma 2.7. *Let (A', B') be an irreducible nonextendible sur-critical pair satisfying* (QP) *of Theorem* 1.4 *with decomposition $A' = A'_1 \cup A'_0$, $B' = B'_1 \cup B'_0$, such that $m(A'_0) > 0$ and $m(B'_0) > 0$. Assume also that $A' + B'$ is aperiodic. If (A, B) is another irreducible nonextendible sur-critical pair such that $A \sim A'$ and $B \sim B'$, then (A, B) also satisfies* (QP)*.*

Proof. Let H be the quasi-period of (A', B'). We may assume $A' \subset A$ and $B' \subset B$, since we can replace A' and B' with $A' \cap A$ and $B' \cap B$ while maintaining the hypotheses of the lemma – this follows from the irreducibility of (A, B) and (A', B'). Assume, without loss of generality, that $A'_1 \neq \emptyset$. Define $A_0 := A \cap (A'_0 + H)$ and $A_1 := A \setminus A_0$. We will show that $A = A_1 \cup A_0$ is a quasi-periodic decomposition of A.

Let $a \in A$, and observe that $a + B' \subset_m A' + B'$, since $A + B \sim A' + B'$. If $a + B'_0 \subset A'_0 + B'_0 + H$, then $a \in A'_0 + H$.

If $a + B'_0 \cap (A'_0 + B'_0 + H) = \emptyset$, then $a + B'_0 \subset_m (A'_1 + B') \cup (A' + B'_1)$. It follows that $a + B' + H \subset_m (A'_1 + B') \cup (A' + B'_1)$. By nonextendibility of (A', B'), we have $a + H \subset_m A' + H$. It follows that $A_1 + H \sim A'_1 + H$. Consequently, $A_1 \cup A_0$ is a quasi-periodic decomposition.

A similar argument shows that B has a quasi-periodic decomposition with respect to H. □

Lemma 2.8. *If Theorem* 1.4 *holds under the additional assumption that $A + B$ is measurable, then Theorem* 1.4 *holds in general.*

Proof. Let G be a compact abelian group and (A, B) a sur-critical pair for G. If (A, B) is extendible, then (A, B) satisfies conclusion (E) of Theorem 1.4. If (A, B) is nonextendible and $A + B$ is periodic, then (A, B) satisfies conclusion (P) of Theorem 1.4. If (A, B) is nonextendible and reducible and $A + B$ is aperiodic, then (A, B) satisfies conclusion (QP) of Theorem 1.4, by Corollary 2.5. We assume for now that (A, B) is nonextendible and irreducible. Let $A' \subset A$ and $B' \subset B$ be countable unions of compact sets with $m(A') = m(A), m(B') = m(B)$, so that $A' + B'$ is measurable.

Irreducibility of (A, B) implies $m(A' + B') = m_*(A + B)$, so (A', B') is a sur-critical pair. If (A', B') satisfies conclusion (P) or conclusion (E) of Theorem 1.4, then $m\big(H(A' + B')\big) > 0$. We apply Lemma 2.6 and conclude that (A, B) satisfies conclusion (P) or (QP). If (A', B') satisfies (K), so that $A' \sim a + \chi^{-1}(I)$ and $B' \sim b + \chi^{-1}(J)$, where $I, J \subset \mathbb{T}$ are intervals and $\chi : H \to \mathbb{T}$ is a continuous homomorphism from a compact open subgroup $H \leq G$, then A and B must be contained in $a + \chi^{-1}(I), b + \chi^{-1}(J)$, respectively. We conclude that (A, B) satisfies (K).

If $A + B$ is not measurable, the only remaining possibility is that (A, B) is irreducible and nonextendible, and (A', B') satisfies (QP): there exists a compact

open subgroup $H \leq G$ and partitions $A' = A'_1 \cup A'_0$, $B' = B'_1 \cup B'_0$, with $A'_1 \sim A'_1 + H$, $B'_1 \sim B'_1 + H$, and $m(A'_0 + B'_0) = m(A'_0) + m(B'_0)$.

If the pair (A'_0, B'_0) satisfies one of (P), (E), or (K), then (A, B) satisfies the conclusion of Theorem 1.4, by the arguments of the first two paragraphs of this proof.

If (A', B') has quasi-periodic decompositions with respect to compact open subgroups of arbitrarily small positive measure, then so does (A, B), by Lemma 2.7. Then $A + B$ is measurable, by Lemma 1.3.

The only remaining possibility is that (A', B') has a quasi-periodic decomposition $A' = A'_1 \cup A'_0$, $B' = B'_1 \cup B'_0$, with quasi-period H, one of $m(A'_0) = 0$ or $m(B'_0) = 0$, and $A'_0 + B'_0 + H$ is a unique expression element of $A' + B' + H$ in G/H. If one of $m(A'_0) > 0$ or $m(B'_0) > 0$, then (A, B) is reducible, contradicting our assumptions. If both $m(A'_0) = 0$ and $m(B'_0) = 0$, then the irreducibility of (A', B') implies $m\big(H(A' + B')\big) > 0$. Then Lemma 2.6 implies (A, B) satisfies either (P) or (QP). □

The following lemma is standard; we include a proof for completeness.

Lemma 2.9. *If $S \subset G$ is measurable and let*

$$E := \{x \in G : 0 < m_{H(S)}(S - x) < 1 \text{ or } H(S) \cap (S - x) \text{ is not measurable}\}.$$

Then $m(E) = 0$.

Proof. Write H for $H(S)$. Consider the integrals

$$I_1 := \int \int 1_S(x - z) 1_S(x) \, dm_H(z) \, dm(x)$$
$$I_2 := \int \int 1_S(x - z) 1_S(x) \, dm(x) \, dm_H(z).$$

The integral I_1 can be computed as

$$\int \int 1_S(x - z) \, dm_H(z) \, 1_S(x) \, dm(x) = \int m_H(S - x) 1_S(x) \, dm.$$

while $I_2 = \int m\big((S + z) \cap S\big) \, dm_H(z) = m(S)$. By Fubini's theorem, $I_1 = I_2$, which means $\int m_H(S - x) 1_S(x) \, dm = m(S)$. Since $0 \leq m_H(S - x) \leq 1$, the last equation implies $m_H(S - x) = 1$ for m-almost every $x \in S$. Setting

$$F := \{x \in S : m_H(S - x) = 1\},$$

we then have $F + H = F$, and $m(F \triangle S) = 0$. Then

$$m(S) = \int_F m_H(S - x) \, dm(x) + \int_{F^c} m_H(S - x) \, dm(x),$$

so $\int_{F^c} m_H(S - x) \, dm(x) = 0$. The assertion follows. □

Lemma 2.10. *If (A, B) is an irreducible sur-critical pair such that $m(A) > 0$, $m(B) > 0$, and $A + B$ is measurable, then there are measurable sets $A' \subset A$, $B' \subset B$ with $m(A') = m(A)$, $m(B') = m(B)$, and $A' + B' + H(A + B) \sim A' + B'$.*

Proof. Let $H := H(A + B)$, and let

$$S_A := \{x \in G : m_H(A - x) > 0\},$$
$$S_B := \{x \in G : m_H(B - x) > 0\},$$

so that S_A and S_B are measurable sets. By Lemma 2.9, the set

$$E := \{x \in G : 0 < m_H(A+B-x) < 1 \text{ or } H \cap (A+B-x) \text{ is not } m_H\text{-measurable}\}$$

has $m_G(E) = 0$, and $E + H = E$. Let A'' and B'' be countable unions of compact sets such that $A'' \subset A, B'' \subset B$, and $m(A'') = m(A)$, $m(B'') = m(B)$. Let $A' = A'' \cap S_A$ and $B' = B'' \cap S_B$, so that $m(A') = m(A)$ and $m(B') = m(B)$. Then $H \cap (A' + B' - x)$ is measurable for every x, and the irreducibility of (A, B) implies that $m(A' + B') = m(A + B)$. By the definition of S_A, and S_B, we have $m_H(A' + B' - x) \neq 0$ whenever $(A' + B' - x) \cap H \neq \emptyset$. Furthermore, $A' + B' + H \sim A' + B'$, since $m_H(A' + B' - x) = 1$ whenever $H \cap \Big((A' + B' - x) \setminus E\Big) \neq \emptyset$. □

Lemma 2.11. *If Theorem* 1.4 *holds under the additional assumption that* $A + B$ *is measurable and* $H(A + B) = \{0\}$ *then Theorem* 1.4 *holds in general.*

Proof. Let (A, B) be a sur-critical pair for G. By Lemma 2.8 we may assume $A+B$ is measurable. Let $A' \subset A$ and $B' \subset B$ be as in the conclusion of Lemma 2.10, and let $K = H(A + B)$. Since (A, B) is nonextendible and irreducible, $A' + K \sim A', B' + K \sim B'$, and $A' + B' + K \sim A' + B' \sim A + B$.

Claim. The pair $(A'+K, B'+K)$ is an irreducible, nonextendible, sur-critical pair for G/K, and $H(A' + B' + K) = \{0\}$ (viewing $A' + B' + K$ as a subset of G/K).

Proof of Claim. We first show $H(A' + B' + K) = \{0\}$. Supposing otherwise, there exists $z \in G \setminus K$ such that $m_G\big((A' + B' + K)\triangle(A' + B' + K + z)\big) = 0$, contradicting the definition of K.

If $(A' + K, B' + K)$ is extendible, then $H(A' + B' + K) \neq \{0\}$, so by the previous paragraph we see that $(A' + K, B' + K)$ is nonextendible.

If $(A'+K, B'+K)$ is reducible, there exists $A'' \subset A'+K, B'' \subset B'+K$ such that $m(A'' + K) = m(A' + K)$, $m(B'' + K) = m(B' + K)$, and $m(A'' + K + B'' + K) < m(A' + K + B' + K)$. Setting $C = A \cap (A'' + K)$, $D = B \cap (B'' + K)$, we find that $m(C) = m(A)$, $m(D) = m(B)$, and $m(C+D) < m(A)+m(B)$. This implies (A, B) is reducible, contradicting our assumptions. □

Now we must show that if the pair $(A'+K, B'+K)$ satisfies any of the conclusions (P), (E), (K), or (QP) of Theorem 1.4, then so does the pair (A, B).

If $(A' + K, B' + K)$ satisfies (P) or (E), then $H\big((A' + K) + (B' + K)\big) \neq \{0\}$, contradicting the claim above. This case is vacuous.

If $(A' + K, B' + K)$ satisfies (K), let $\chi' : G/H \to \mathbb{T}$ be the homomorphism of conclusion (K), and let $I, J \subset \mathbb{T}$ be the corresponding intervals. If $\chi = \chi' \circ p$, where $p : G \to G/K$ is the quotient map, it is easy to check that (A, B) satisfies $m(A) = m\big(\chi^{-1}(I)\big)$, $m(B) = m\big(\chi^{-1}(J)\big)$, while $A \subset a + \chi^{-1}(I)$, $B \subset b + \chi^{-1}(J)$ for some $a, b \in G$.

If $(A' + K, B' + K)$ satisfies (QP) but not (P) or (E), let $A' + K = A_1' \cup A_0'$, $B' + K = B_1' \cup B_0'$ be quasi-periodic decompositions of $A' + K$ and $B' + K$ with quasi-period W, such that $A_0'+B_0'$ is a unique expression element of $A'+B'+W+K$ in $G/(W + K)$, and $m_{G/K}(A_0' + K + B_0' + K) = m_{G/K}(A_0' + K) + m_{G/K}(B_0' + K)$ Assume, without loss of generality, that $A_1' \neq \emptyset$. We will show that $m(A_0'+K) > 0$. Assume, to get a contradiction, that $m(A_0' + K) = 0$. Then by irreducibility and nonextendibility of $(A'+K, B'+K)$, either $m(B_0'+K) = 0$ and $m(A_0'+B_0'+K) = 0$,

or $m(B_0'+K)=1$. In both cases, $H(A'+B'+K)$ has positive Haar measure, contradicting the claim above.

Finally, assuming $m(A_0'+K)>0$ and $m(B_0'+K)>0$, Lemma 2.7 implies that (A,B) satisfies (QP). $\square$

2.2. **Quasi-periodicity and complements.** The consequences of Lemmas 2.13 - 2.16 are summarized by the following proposition. The hypothesis $m\big(H(A+B)\big)=0$ will include an implicit assumption that $A+B$ is measurable.

Proposition 2.12. *Let (A,B) be an irreducible, nonextendible, sur-critical pair such that $m\big(H(A+B)\big)=0$. If one of the sets A,B, or $A+B$ is quasi-periodic with respect to a compact open subgroup H, then all of these sets are quasi-periodic with respect to H. If $A=A_1\cup A_0$ and $B=B_1\cup B_0$ are the corresponding quasi-periodic decompositions, then A_0+B_0+H is a unique expression element of $A+B+H$ in G/H, and $m(A_0+B_0)=m(A_0)+m(B_0)$.*

Lemma 2.13 (cf. [2], Lemma 2.4)**.** *If (A,B) is an irreducible nonextendible sur-critical pair with $m(H(A+B))=0$, then*

$$-B+(A+B)^c\sim A^c, \tag{2.9}$$

$$-A+(A+B)^c\sim B^c, \tag{2.10}$$

and both $(-B,(A+B)^c)$ and $(-A,(A+B)^c)$ are irreducible nonextendible sur-critical pairs with $m\big(H(-B+(A+B)^c)\big)=0$ and $m\big(H(-A+(A+B)^c)\big)=0$.

Lemma 2.13 is analogous to Lemma 2.4 of [2]. As shown in [2], $-B+(A+B)^c\subset A^c$ unconditionally.

Proof. Assuming (A,B) is an irreducible nonextendible sur-critical pair such that $m\big(H(A+B)\big)=0$, we first establish (2.9) and (2.10). Let E be the set of $y\in A^c$ such that $y+B\cap(A+B)^c\neq\emptyset$. Let $S=A^c\setminus E$. Then $(S+B)\cap(A+B)^c=\emptyset$. By the nonextendibility of (A,B), we have $m(S\setminus A)=0$. Since $S\subset A^c$, we have $m(S)=0$, which implies $m(E)=m(A^c)$. From the definition of E, we have $E\subset -B+(A+B)^c$. Combining this with the unconditional containment $-B+(A+B)^c\subset A^c$, we obtain (2.9), and consequently (2.10) by symmetry.

Now we must show that $(-A,(A+B)^c)$ is an irreducible sur-critical pair such that $M:=m\big(H(-A+(A+B)^c)\big)=0$. Suppose first that $M>0$, so that $H(B^c)>0$, by (2.10). Then $H(B)>0$, and $B\sim B'\sim B'+H$ for some $B'\subset B$ with $m(B')=m(B)$. By irreducibility of (A,B), we have $A+B\sim A+B'$, which means $m\big(H(A+B)\big)>0$, contradicting our assumptions on (A,B).

Note that

$$\begin{aligned} m(-A)+m\big((A+B)^c\big)&=m(A)+1-m(A+B)\\ &=1-m(B)\\ &=m(B^c), \end{aligned}$$

so $(-A,(A+B)^c)$ is a sur-critical pair by (2.10).

Now suppose, to get a contradiction, that $(-A,(A+B)^c)$ is reducible. By Corollary 2.5, reducibility of $(-A,(A+B)^c)$ yields quasi-periodic decompositions $A=A_1\cup A_0$ and $(A+B)^c=D_1\cup D_0$ with respect to a compact open subgroup $H\leq G$, and one of $m(A_0)=0$ or $m(D_0)=0$. If $m(A_0)=0$, the irreducibility of

(A, B) implies $H(A + B) > 0$, contradicting the hypothesis. If $m(A_0) > 0$, then $m(D_0) = 0$, and $H\big((A + B)^c\big) > 0$. Then $H := H(A + B) > 0$, contradicting the hypothesis. We have shown that $(-A, (A+B)^c)$ is irreducible, and the irreducibility of $(-B, (A + B)^c)$ follows similarly.

To show that $(-A, (A+B)^c)$ is nonextendible, assume otherwise to find a contradiction. Then by Theorem 1.1, $m\big(H(-A+(A+B)^c)\big) > 0$. Since $-A+(A+B)^c \sim B^c$ by previous arguments, we have $m\big(H(B^c)\big) > 0$, so $m\big(H(B)\big) > 0$. By the irreducibility of (A, B), this implies $m\big(H(A+B)\big) > 0$, which is the desired contradiction.

We have shown that $(-A, (A + B)^c)$ is an irreducible sur-critical pair satisfying $m\big(H(-A+(A+B)^c)\big) > 0$. Reversing the roles of A and B yields the corresponding description of $(-B, (A + B)^c)$. □

Lemma 2.14 (cf. [2], Lemma 5.1)**.** *Let (A, B) be a sur-critical pair with $m(A) > 0$ and $m(B) > 0$, and let H be the subgroup of G generated by A (so H is compact and open). If $A + B$ is aperiodic and B is nonextendible with respect to A, then B has a quasi-periodic decomposition with respect to H, or B is contained in a coset of H.*

Proof. Let B' be the maximal H-periodic subset of B, so that $B' \sim B' + H$, and $m(B \cap H_i) < m(H)$ for all cosets H_i of H disjoint from $B' + H$. Let $B \setminus B' = B_1 \cup \cdots \cup B_l$ be an H-coset decomposition of $B \setminus B'$, so that no B_i is H-periodic, and $b_i \in B_i$ for each i. We aim to show that $l = 1$, implying B is quasi-periodic.

Since B is nonextendible, it follows that $m\big((b_i + H) \setminus (A + B_i)\big) > 0$ for all i. Since $A \subset H$, we have

$$m(A + B) = m(B') + m\big(A + (B \setminus B')\big),$$

while

$$m(A + B) = m(A) + m(B) = m(A) + m(B') + m(B \setminus B').$$

Hence $m\big(A + (B \setminus B')\big) = m(A) + m(B \setminus B')$. We assume, to get a contradiction, that $l \geq 2$. Then

$$\sum_i m(A + B_i) = m(A) + \sum_i m(B_i),$$

so that

$$\sum_i m(A + B_i) - m(B_i) = m(A).$$

If $m(A + B_i) < m(A) + m(B_i)$ for all i, then by Theorem 1.1, $A + B$ is periodic, contradicting our hypothesis. If $m(A + B_i) = m(A) + m(B_i)$ for some i, then $m(A + B_j) = m(B_j)$ for $j \neq i$. Consequently, $A \subset H(B_j)$ and $m(B_j) > 0$ for all $j \neq i$. This implies $H \leq H(B_j)$ and $B_j \sim B_j + H$ for all $j \neq i$, contradicting the maximality of B'. □

Lemma 2.15 (cf. [2], Lemma 5.3)**.** *Let (A, B) be an irreducible nonextendible sur-critical pair with $m\big(H(A + B)\big) = 0$, and let $A = A_1 \cup A_0$ be a quasi-periodic decomposition with A_1 nonempty and periodic with maximal period H. Then B has a quasi-periodic decomposition $B = B_1 \cup B_0$ with quasi-period H, such that:*

(i) $A_0 + B_0 + H$ *is a unique expression element of* $A + B + H$ *in* G/H, *and*

(ii) $m(A_0 + B_0) = m(A_0) + m(B_0)$.

So (A, B) *satisfies the conclusion* (QP) *of Theorem* 1.4.

Proof. First, observe that $m(A_1 + B) < m(A + B)$, since $m\big(H(A_1 + B)\big) > 0$, while $m\big(H(A + B)\big) = 0$. This implies

$$\begin{aligned} m(A_1 + B) &\leq m(A + B) - m(A_0) \\ &= m(A) + m(B) - m(A_0) \\ &= m(A_1) + m(B). \end{aligned} \tag{2.11}$$

Note that $m(A_0) > 0$, by irreducibility of (A, B).

Claim. Strict inequality holds in (2.11).

Proof of Claim. Assuming $m(A_1 + B) = m(A + B) - m(A_0)$, then $m(A + B) = m(A_1 + B) + m(A_0)$, so $A + B \sim (A_1 + B) \cup (A_0 + b_0)$ for some $b_0 \in B$. Consequently, the set

$$B_0 := \{b \in B : (A + b) \setminus (A_1 + B) \neq \emptyset\}$$

is contained in a coset of $H(A_0)$. If $m\big(H(A_0)\big) = 0$, then $m(B_0) = 0$ and (A, B) is reducible, contrary to the hypothesis. If $m\big(H(A_0)\big) > 0$, the irreducibility of (A, B) implies $m\big(H(A + B)\big) > 0$, again contradicting our hypothesis. □

By Theorem 1.1 applied to (A_1, B), there exists a compact open subgroup $H' \leqslant G$ such that $A_1 + B = A_1 + B + H'$, and

$$m(A_1 + B) = m(A_1 + H') + m(B + H') - m(H'). \tag{2.12}$$

Now $A_1 + B + H' \subset A_1 + B$, so $\big((A_1 + B + H') \cup (A_0 + B)\big) \subset A + B$. Thus $A_1 + H' \sim A_1$, by nonextendibility of A. Now $H' \leqslant H$, by maximality of H, and $H \leqslant H'$ by Theorem 1.1, so $H' = H$.

Let $b \in B$ such that $A_0 + b \subset (A_1 + B)^c$, and let $B_0 = (b + H) \cap B$, $B_1 = B \setminus B_0$. We aim to show that $B_1 \cup B_0$ is a quasi-periodic decomposition of B. Lemma 2.1, applied to (A_1, B), implies $m(B_0) + m(A_1 + B) \geq m(A_1) + m(B)$. Then either

$$m(A_0 + B_0) < m(A_0) + m(B_0), \tag{2.13}$$

or

$$\begin{aligned} m(A + B) &\geq m(A_0 + B_0) + m(A_1 + B) \\ &\geq m(A_0) + m(B_0) + m(A_1 + B) \\ &= m(A_0) + m(B_0) + m(A_1 + H) + m(B + H) - m(H) \\ &\geq m(A) + m(B). \end{aligned} \tag{2.14}$$

If (2.13) holds for all $b \in B$ such that $A_0 + b \subset (A_1 + B)^c$, then $m\big(H(A + B)\big) > 0$, by Theorem 1.1, contradicting the hypothesis. Therefore, we assume (2.14) holds for some choice of $b \in B$ and show that $B = B_1 \cup B_0$ is a quasi-periodic decomposition. What remains to be shown is that

$$B_1 + H \sim B_1. \tag{2.15}$$

Now (2.14) implies

$$A + B \sim (A_0 + B_0) \cup (A_1 + B), \tag{2.16}$$

which is a disjoint union. It follows that $A_0 + B_0 + H$ is a unique expression element of $A+B+H$ in G/H, since otherwise there exists $a \in A_1$ and $b' \in B$ such that $a + b' + H \subset_m (A_0 + B_0 + H) \cap (A + B)$, contradicting the assumption that $m(H(A+B)) = 0$. Since A_0+B_0+H is a unique expression element of $A+B+H$ in G/H, (2.16) implies $B_1 + A \subset A_1 + B + H \sim A_1 + B$. The nonextendibility of (A, B) and (2.15) then imply $B_1 \sim B_1 + H$. This concludes the proof of (i).

Statement (ii) of the lemma follows from two observations. First, observe that if $m(A_0 + B_0) < m(A_0) + m(B_0)$, then Theorem 1.1 implies $A + B$ is periodic, contradicting the hypothesis that $m\big(H(A + \ B)\big) = 0$. Second, observe that if $m(A_0 + B_0) > m(A_0) + m(B_0)$, then

$$\begin{aligned} m(A+B) &= m(A_1 + B) + m(A_0 + B_0) \\ &= m(A_1 + H) + m(B + H) - m(H) + m(A_0 + B_0) \qquad \text{(by (2.12))} \\ &> m(A_1) + m(A_0) + m(B + H) - m(H) + m(B_0) \\ &= m(A) + m(B), \end{aligned}$$

contradicting the hypothesis that (A, B) is a sur-critical pair. □

Lemma 2.16. *Let (A, B) be an irreducible nonextendible sur-critical pair for G such that $m\big(H(A+B)\big) = 0$.*

(i) *If $A+B$ is quasi-periodic with respect to a compact open subgroup $H \leqslant G$, then (A, B) satisfies conclusion* (QP) *of Theorem* 1.4.
(ii) *If $A + B \sim D$, where D is quasi-periodic with respect to a compact open subgroup $H \leq G$, then (A, B) satisfies conclusion* (QP) *of Theorem* 1.4.

Proof. (i) If A is contained in a coset of H, we may apply Lemma 2.14 to conclude that B has a quasi-periodic decomposition with respect to some compact open $H' \leqslant H$. Otherwise, write C for $(A+B)^c$, and note that the assumption that $A+B$ is quasi-periodic with quasi-period H implies $C \sim C' \subset C$, where C' is quasi-periodic with respect to H. Lemma 2.13 implies $\big(-B, C\big)$ is an irreducible nonextendible sur-critical pair with $m\big(H(-B + C)\big) = 0$ and $-B + C \sim A^c$. Irreducibility of $(-B, C)$ implies $-B + C' \sim -B + C \sim A^c$. Lemma 2.15 then implies that $-B$ is quasi-periodic, and the same lemma applied to (A, B) gives a quasi-periodic decomposition of (A, B) satisfying (QP).

(ii) If $A + B$ is not quasi-periodic with respect to H, then there is at least one coset $H_1 = a_1 + b_1 + H$, which meets $A + B$ and has $m\big(H_1 \cap (A+B)\big) = 0$. Let W be the union of all such cosets, and let

$$\begin{aligned} \tilde{A} &:= \{a \in A : \text{ there exists } b \in B \text{ such that } a + b \in W\} \\ \tilde{B} &:= \{b \in B : \text{ there exists } a \in A \text{ such that } a + b \in W\}. \end{aligned}$$

Then $\tilde{A}$ and $\tilde{B}$ have measure 0. Let $A' = A \setminus \tilde{A}, B' = B \setminus \tilde{B}$. If $m(A' + B') < m(A)+m(B)$, then (A, B) is reducible, contradicting our assumptions. We conclude that $A' + B'$ is quasi-periodic, and Part (i) yields a quasi-periodic decomposition $A' = A'_1 \cup A'_0$, $B' = B'_1 \cup B'_0$ of (A', B') satisfying (QP). Since (A, B) is irreducible and $m\big(H(A + B)\big) = 0$, we have $m(A'_0) > 0$ and $m(B'_0) > 0$. Now Lemma 2.7 implies that (A, B) also has a quasi-periodic decomposition satisfying (QP). □

2.3. **The e-transform.** For $A, B \subset G$ and $e \in G$, let

$$A_e := A \cap (B + e)$$
$$B_e := (A - e) \cap B.$$

This is the classical e-transform, whose properties are well-documented. In particular, $A_e + B_e \subset A + B$, and $m(A_e) + m(B_e) = m(A) + m(B)$ whenever B_e is nonempty. See [8] or [12] for further exposition.

Lemma 2.17. *Let (A, B) be a sur-critical pair for G such that $A+B$ is measurable. If $H(A + B) = \{0\}$, then one of the following holds.*

(i) *There exists $e \in G$ such that $0 < m(B_e) \leq \big(1 - m(B)\big)m(B)$.*

(ii) *There exists $e \in G$ such that $B_e = \{0\}$.*

(iii) $m(A - B) < m(A) + m(B)$.

Proof. Write $f(z) = m\big((A-z)\cap B\big)$, so that $f : G \to [0, 1]$ is a continuous function and $\int f\, dm = m(A)m(B)$. Consider $S := \{z : f(z) > 0\}$, which is contained in $A - B$. If $m(S) < m(A - B)$, then there is an $e \in A - B$ with $m(B_e) = 0$. In this case, $m(A_e) = m(A+B)$, and since $A_e+B_e \subset A+B$, it follows that $B_e \subset H(A+B)$. Thus, $B_e = \{0\}$, and we conclude (ii).

If $m(S) = m(A - B)$, we form the average

$$\frac{1}{m(S)} \int_S f dm = \frac{1}{m(A - B)} m(A)m(B). \tag{2.17}$$

When $m(A - B) \geq m(A) + m(B)$, equation (2.17) implies

$$\frac{1}{m(S)} \int_S f dm \leq \big(1 - m(B)\big)m(B),$$

and we conclude (i). Otherwise, we conclude (iii). □

2.4. **The key lemma.** In the next lemma, we use $\phi_H : G \to G/H$ to denote the quotient map. For $A \subset G$, $\#\phi_H(A)$ is the cardinality of $\phi_H(A)$, that is, the number of cosets of H having nonempty intersection with A.

Lemma 2.18 (cf. [2], §6 ("Subcase 1"))**.** *Suppose (A, B) is an irreducible nonextendible sur-critical pair with $m\big(H(A + B)\big) = 0$, and there exists $e \in G$ such that $m(A_e + B_e) < m(A) + m(B)$. Then (A, B) satisfies conclusion* (QP) *of Theorem* 1.4.

Proof. Without loss of generality, assume $e = 0$, so that $A_e = A \cup B$, and $B_e = A \cap B$. Applying Theorem 1.1 to (A_e, B_e) yields a compact open subgroup H with $A_e + B_e = A_e + B_e + H$, and $m(A_e + B_e) = m(A_e + H) + m(B_e + H) - m(H)$. Let

$$\rho := m\big((A_e + H) \setminus A_e\big) + m\big((B_e + H) \setminus B_e\big),$$
$$\rho' := m\big((A + H) \setminus A\big) + m\big((B + H) \setminus B\big).$$

Partition A as $A = A_0 \cup A_1 \cup A_2$, where $A_0 = \big(A \cap (B + H)\big) \setminus (A \cap B)$, $A_1 = A \cap B$, and $A_2 = A \setminus (A_0 \cup A_1)$, and partition B similarly, as $B_0 = \big(B \cap (A+H)\big) \setminus (A \cap B)$, $B_1 = A \cap B$, and $B_2 = B \setminus (B_0 \cup B_1)$.

Claim 1. $(A \cap B) + H \sim A \cap B$.

Proof of claim. If $b \in A \cap B$, then $A + b \subset A_e + B_e = A_e + B_e + H$, so $A + b + H \subset A + B$. Since (A, B) is nonextendible, it follows that $b + H \subset_m B$. Likewise, if $a \in A \cap B$, then $a + H$ is essentially contained in A. The claim follows. □

Claim 2. $A_0 + H = B_0 + H$.

Proof of claim. The definitions of A_0 and B_0 imply $A + H$ contains B_0, while $A_1 + H$ and $A_2 + H$ are disjoint from B_0, and $B_1 + H$ and $B_2 + H$ are disjoint from A_0. Thus $B_0 \subset A_0 + H$ and $A_0 \subset B_0 + H$. □

Let $\rho'' := m\big((A_2 + H) \setminus A_2\big) + m\big((B_2 + H) \setminus B_2\big)$. Now $A_0 \cap B_0 = \emptyset$, so the second claim implies

$$m(A_0 \cup B_0) = m(A_0) + m(B_0) \leq \#\phi_H(A_0) \cdot m(H).$$

Then

$$\rho = \rho'' + \#\phi_H(A_0) \cdot m(H) - m(A_0) - m(B_0) \geq \rho'', \tag{2.18}$$

while

$$\begin{aligned} m(A_e + B_e) &= m(A_e + H) + m(B_e + H) - m(H) \\ &\geq m(A) + m(B) - m(H) + \rho. \end{aligned} \tag{2.19}$$

We now consider two cases.

Case 1. There exists $b \in B_2$ with $\#\big[\phi_H(A + b) \setminus \phi_H(A_e + B_e)\big] = t \geq 1$. In this case, $\rho'' \geq m(H) - m\big(B \cap (b + H)\big)$. Then (2.18), (2.19), and the definition of H imply

$$\begin{aligned} m(A + B) &\geq m(A_e + B_e) + t \cdot m\big(B \cap (b + H)\big) \\ &= m(A_e + H) + m(B_e + H) - m(H) + t \cdot m\big(B \cap (b + H)\big) \\ &\geq m(A) + m(B) - m(H) + \rho'' + m\big(B \cap (b + H)\big) \\ &\geq m(A) + m(B), \end{aligned} \tag{2.20}$$

so the inequalities are actually equations. Note that $(B \cap (b+H)) = A_e \cap (b+H)$, by the definition of B_2 and the second claim. Lemma 2.1, applied to (A_e, B_e), implies

$$m(A_e + B_e) + m\big(B \cap (b + H)\big) \geq m(A) + m(B),$$

so $t = 1$, and

$$m(A + B) = m(A_e + B_e) + m\big(B \cap (b + H)\big).$$

Thus

$$A + B \sim (A_e + B_e) \cup [a + (B \cap (b + H))] \text{ for some } a \in A. \tag{2.21}$$

By Part (ii) of Lemma 2.16, (2.21) implies (A, B) satisfies conclusion (QP) of Theorem 1.4.

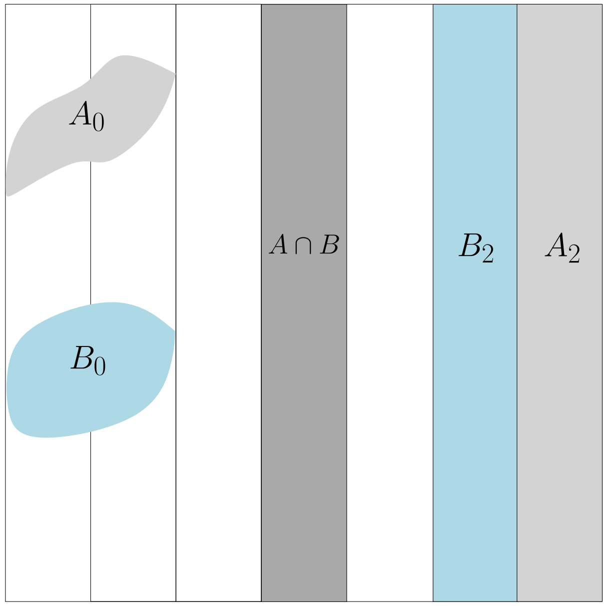

FIGURE 1.

Case 2. For all $b \in B_2$, $A+b \subset A_e+B_e$. We assume also that for all $a \in A_2, a+B \subset A_e + B_e$, or else we could repeat the argument of Case 1 with the roles of A and B reversed. Since (A, B) is nonextendible, we get that $b + H \subset_m B$ for all $b \in B_2$, and $a + H \subset_m A$ for all $a \in A_2$.

We now have $A = A_0 \cup A_1 \cup A_2$, where $A_1 + H \sim A_1, A_2 + H \sim A_2$, and $A_0 \cap B = \emptyset$. Similarly, $B = B_0 \cup B_1 \cup B_2$, where $B_1 + H = B_1, B_2 + H = B_2$, and $B_0 \cap A = \emptyset$. Also, $A_0 + H = B_0 + H$. This situation is depicted in Figure 1: the vertical rectangles are cosets of H, while contra the conclusion, A_0 and B_0 are shown as occupying two cosets of H. Our immediate goal is to show that $A + B$ has a quasi-periodic decomposition with quasi-period H.

Let

$$A_0 = \bigcup_{i=1}^{l} A_{0,i}, \ B_0 = \bigcup_{i=1}^{l} B_{0,i}$$

be H-coset decompositions of A_0 and B_0, with $A_{0,i} + H = B_{0,i} + H =: H_i$ for all i. Then $A_e \cap H_i = (A_{0,i} \cup B_{0,i})$, which is a disjoint union, for each i. Lemma 2.1, applied to (A_e, B_e), implies

$$m(A_{0,i}) + m(B_{0,i}) + m(A_e + B_e) \geq m(A_e) + m(B_e) = m(A) + m(B), \tag{2.22}$$

for all i.

Assume, to get a contradiction, that $A+B$ is not quasi-periodic with quasi-period H. Then there are two pairs (i, j), (i', j') such that the sets $(A_{0,i}+B_{0,j}) \cup (A_{0,j}+B_{0,i})$ and $(A_{0,i'}+B_{0,j'}) \cup (A_{0,j'}+B_{0,i'})$ occupy distinct cosets of H and are disjoint from $(A_e + B_e)$.

Writing

$$\begin{aligned} M &:= \max\{m(A_{0,i}), m(A_{0,j}), m(B_{0,i}), m(B_{0,j})\} \\ M' &:= \max\{m(A_{0,i'}), m(A_{0,j'}), m(B_{0,i'}), m(B_{0,j'})\}, \end{aligned}$$

we estimate $m(A+B)$ from below by

$$m(A+B) \geq M + M' + m(A_e + B_e). \tag{2.23}$$

Combining (2.23) with (2.22) yields

$$M + M' \leq m(A_{0,s}) + m(B_{0,s}) \text{ for all } s \in \{i, i', j, j'\}, \tag{2.24}$$

so in particular, $M + M' \leq \min_s\{m(A_{0,s}) + m(B_{0,s})\}$. Assume $M \leq M'$. Since

$$\min_s\{m(A_{0,s}) + m(B_{0,s})\} \leq \min_s \min\{m(A_{0,s}), m(B_{0,s})\} + M',$$

(2.24) implies

$$M = \min_s \min\{m(A_{0,s}), m(B_{0,s})\}. \tag{2.25}$$

From the definition of M, (2.25) implies

$$\begin{aligned} M &= m(A_{0,i}) = m(A_{0,j}) = m(B_{0,i}) = m(B_{0,j}) \\ &= \min\{m(A_{0,i'}), m(A_{0,j'}), m(B_{0,i'}), m(B_{0,j'})\}. \end{aligned} \tag{2.26}$$

Combining (2.26) with (2.24) we conclude $M = M'$, and

$$m(A_{0,s}) = m(A_{0,t}) = m(B_{0,s}) = m(B_{0,t}) \text{ for all } s, t \in \{i, j, i', j'\}. \tag{2.27}$$

Again estimating $m(A+B)$, the assumption that $A+B$ is not quasi-periodic implies

$$\begin{aligned} m(A+B) &\geq m(A_e + B_e) + m\big((A_{0,i} + B_{0,j}) \cup (A_{0,j} + B_{0,i})\big) \\ &\qquad + m\big((A_{0,i'} + B_{0,j'}) \cup (A_{0,j'} + B_{0,i'})\big) \\ &\geq m(A_e + B_e) + \max\{m(A_{0,i} + B_{0,j}), m(A_{0,j} + B_{0,i})\} \\ &\qquad + \max\{m(A_{0,i'} + B_{0,j'}), m(A_{0,j'} + B_{0,i'})\}. \end{aligned} \tag{2.28}$$

Assuming $A+B$ is aperiodic, Theorem 1.1 implies one of

$$m(A_{0,i} + B_{0,j}) \geq m(A_{0,i}) + m(B_{0,j}),$$

or another such relation with permuted indices holds. Then the last line in (2.28) is at least

(2.29)

$$\begin{aligned} & m(A_e + B_e) + m(A_{0,i}) + m(B_{0,j}) + \max\{m(A_{0,i'} + B_{0,j'}), m(A_{0,j'} + B_{0,i'})\} \\ &\quad = m(A_e + B_e) + m(A_{0,i}) + m(B_{0,i}) + \max\{m(A_{0,i'} + B_{0,j'}), m(A_{0,j'} + B_{0,i'})\} \\ &\quad > m(A+B), \end{aligned}$$

where the first equation is a consequence of (2.27), and the last inequality follows from Lemma 2.1 applies to (A_e, B_e), using the fact that

$$m(A_{0,i}) + m(B_{0,i}) = m\big((a+H) \cap A_e\big)$$

for some $a \in A_e$. Inequalities (2.28) and (2.29) imply $m(A+B) > m(A+B)$, which is the desired contradiction. We are done with Case 2. □

2.5. **Miscellaneous lemmas.**

Lemma 2.19. *Suppose $A, B \subset G$ such that $m(A) > 0, m(B) > 0$, and $m(A-B) < m(A) + m(B)$. Then $A + B$ is periodic.*

Proof. Let H be the compact open subgroup given by Theorem 1.1, where $m(A - B) = m(A + H) + m(B + H) - m(H)$. Observe that

$$m\big((a + H) \cap A\big) + m\big((b + H) \cap B\big) > m(H) \tag{2.30}$$

for all $a \in A, b \in B$, since otherwise $m(A+H)+m(B+H)-m(H) \geq m(A)+m(B)$. Now (2.30) implies $\big((a + H) \cap A\big) + \big((b + H) \cap B\big) = a + b + H$, for all $a \in A, b \in B$, so $A + B = A + B + H$. □

Lemma 2.20. *If $A, B \subset G$ and $m(A), m(B) > 0$, there are sets $A' \subset A, B' \subset B$ with $m(A') = m(A), m(B') = m(B)$, and $m\big(A' \cap (B'+t)\big) > 0$ whenever $A' \cap (B'+t)$ is nonempty.*

Proof. By [10], Theorem A, there is a sequence of neighborhoods U_n of the identity $0 \in G$ such that for m-almost all $a \in A, b \in B$,

$$\lim_{n\to\infty} \frac{m\big(A \cap (U_n + a)\big)}{m(U_n)} = 1, \lim_{n\to\infty} \frac{m\big(B \cap (U_n + b)\big)}{m(U_n)} = 1. \tag{2.31}$$

Let $A' \subset A$ and $B' \subset B$ be the sets of points in A and B, respectively, satisfying (2.31), so that $m(A') = m(A)$ and $m(B') = m(B)$. If $t \in G$ and $A' \cap (B' + t)$ is nonempty, assume without loss of generality that $0 \in A' \cap (B' + t)$. Now (2.31) implies that for sufficiently large n, we have

$$m\big(A' \cap U_n\big) > .6m(U_n) \text{ and } m\big((B' + t) \cap U_n\big) > .6m(U_n),$$

so $m(A' \cap (B' + t) \cap U_n) > .1m(U_n)$. □

Lemma 2.21 ([8], Lemma 6)**.** *If $A, B \subset G$ are measurable and $m(A + B) \leq m(A) + \delta$, then for all $x, y \in B$ and $n \in \mathbb{N}$,*

$$m\Big(\bigcup_{i=0}^{n} (A + B) + i(y - x)\Big) \leq m(A + B) + n\delta.$$

□

3. The main argument

3.1. **The sequence of e-transforms.**

We now fix an *infinite* compact abelian group G.

Lemma 3.1. *Let (A, B) be an irreducible, nonextendible, sur-critical pair for G such that $H(A + B) = \{0\}$. Then one of the following holds.*

(i) *There is a sequence of pairs $(A^{(n)}, B^{(n)}), n = 0, 1, 2, \ldots$ such that $A^{(0)} = A$, $B^{(0)} = B$, and for all n,*
   - (i.1) *the pair $(A^{(n+1)}, B^{(n+1)})$ is derived from $(A^{(n)}, B^{(n)})$ by e-transform,*
   - (i.2) *$A^{(n+1)} + B^{(n+1)} \sim A + B$, and*
   - (i.3) *$0 < m(B^{(n+1)}) \leq \big(1 - m(B^{(n)})\big)m(B^{(n)})$,*

   so that $\lim_{n\to\infty} m(B^{(n)}) = 0$.

(ii) *(A, B) satisfies conclusion (QP) of Theorem 1.4.*

Proof. Suppose $(A^{(l)}, B^{(l)})$ satisfies the description in (i) for $l = 0, \dots, n$. We will show that either there is a pair $(A^{(n+1)}, B^{(n+1)})$ satisfying (i.1)-(i.3), or (ii) holds.

First, note that $(A^{(n)}, B^{(n)})$ is not extendible, since $H(A^{(n)} + B^{(n)}) = \{0\}$.

If the pair $(A^{(n)}, B^{(n)})$ is reducible, then $n > 0$, and Corollary 2.5 says that $(A^{(n)}, B^{(n)})$ has a quasi-periodic decomposition. Since $A + B \sim A^{(n)} + B^{(n)}$, Part (ii) of Lemma 2.16 implies (A, B) satisfies conclusion (QP) of Theorem 1.4.

If $(A^{(n)}, B^{(n)})$ is irreducible, apply Lemma 2.20 to replace the pair $(A^{(n)}, B^{(n)})$ with a pair (C, D) where $C \subset A^{(n)}, D \subset B^{(n)}$, $m(C)+m(D) = m(A^{(n)})+m(B^{(n)})$, and $(C-e)\cap D \neq \emptyset$ if and only if $m\big((C-e)\cap D\big) > 0$. By Lemma 2.17, one of the following holds: there exists $e \in G$ with

$$0 < m(D_e) < \big(1 - m(D)\big)m(D) \tag{3.1}$$

or $m(C-D) < m(C)+m(D)$. When $m(C-D) < m(C)+m(D)$, Lemma 2.19 says that $C + D$ is periodic. Thus $H(A^{(n)} + B^{(n)}) \neq \{0\}$, implying $H(A + B) \neq \{0\}$, contradicting our assumptions.

If we find e satisfying (3.1), take $B^{(n+1)} = (B^{(n)})_e, A^{(n+1)} = (B^{(n)})_e$. If the equation $m(A^{(n+1)} + B^{(n+1)}) = m(A)+m(B)$ holds, we conclude (i.1)-(i.3), since $B^{(n)} \sim D$. Otherwise, $m(A^{(n+1)} + B^{(n+1)}) < m(A^{(n+1)})+m(B^{(n+1)})$, and Lemma 2.18 implies $A^{(n)} + B^{(n)}$ $(\sim A+B)$ is quasi-periodic. Now Part (ii) of Lemma 2.16 implies (A, B) satisfies conclusion (QP) of Theorem 1.4. We conclude (ii).

If the construction of $(A^{(n)}, B^{(n)})$ yields conclusion (ii) for some n, we are done. Otherwise, we have (i). □

We now prove a special case of Theorem 1.4.

Proposition 3.2. *If (A, B) is an irreducible, nonextendible, sur-critical pair for G such that $H(A + B) = \{0\}$, then (A, B) satisfies the conclusion of Theorem* 1.4.

Proof. The hypotheses of Lemma 3.1 are satisfied. If (ii) holds in the conclusion of Lemma 3.1, the conclusion of Theorem 1.4 is true. Now we must analyze the case where the conclusion (i) holds in Lemma 3.1. We follow the arguments of [8], §4.

For the following sequence of claims, fix an irreducible nonextendible sur-critical pair for G satisfying $H(A + B) = \{0\}$, and and fix a sequence $(A^{(n)}, B^{(n)})$ of pairs satisfying (i) of Lemma 3.1.

Claim 1. For every neighborhood U of $0 \in G$, there exists n with $B^{(n)} - B^{(n)} \subset U$.

Proof of Claim 1. Suppose not. Then, since $B^{(n+1)} \subset B^{(n)}$ for all n, there exists a neighborhood U of 0 and elements $x_n, y_n \in B^{(n)}$ such that $x_n - y_n \notin U$ for all $n \in \mathbb{N}$. Since G is compact, the sequences x_n, y_n have limit points x, y, with $x - y \neq 0$. Note that $A^{(n)} \subset (A^{(n)} + B^{(n)} - y_n) \cap (A^{(n)} + B^{(n)} - x_n)$. Then

$$\begin{aligned}
m\big((A+B)\cap(A+B+y-x)\big) &\geq \liminf_{n\to\infty} m\big((A+B)\cap(A+B+y_n-x_n)\big)\\
&= \liminf_{n\to\infty} m\big((A^{(n)}+B^{(n)}-y_n)\cap(A^{(n)}+B^{(n)}-x_n)\big)\\
&\geq \lim_{n\to\infty} m(A^{(n)})\\
&= m(A)+m(B)\\
&= m(A+B),
\end{aligned}$$

so $A+B+(y-x) \sim A+B$, contradicting the assumption $H(A+B)=\{0\}$. $\square$

We now consider two separate cases.

Case 1. G is totally disconnected.

Claim 2. If G is totally disconnected, then (A,B) satisfies (QP).

Proof of Claim 2. Since G is totally disconnected, there is a neighborhood base of the identity $0 \in G$ consisting of compact open subgroups. Fix a compact open subgroup K and $n \in \mathbb{N}$ such that $B^{(n)} - B^{(n)} \subset K$, while $m(A^{(n)}) > m(K)$. Then Lemma 2.14 implies $A^{(n)}$ is quasi-periodic with quasi-period $K' \leq K$, and Lemma 2.15 implies $A^{(n)} + B^{(n)}$ is quasi-periodic with quasi-period K'. Since $A+B \sim A^{(n)} + B^{(n)}$, Part (ii) of Lemma 2.16 implies (A,B) satisfies (QP). $\square$

Case 2. G is not totally disconnected. If G is not totally disconnected, let $\chi : G \to \mathbb{T}$ be a surjective continuous homomorphism. By Claim 1, there is a sequence of closed intervals $[-a_n, a_n] \subset \mathbb{T}$ with lengths $2a_n$ such that $B^{(n)} - B^{(n)} \subset \chi^{-1}([-a_n, a_n])$ for all n, and $\lim_{n\to\infty} a_n = 0$. Assuming each a_n is the least $s > 0$ with $B^{(n)} - B^{(n)}$ contained in $\chi^{-1}([-s,s])$, we have $a_n = \chi(x_n) - \chi(y_n)$ for some x_n, y_n in the closure of $B^{(n)}$. Adjoining the points x_n, y_n to the set $B^{(n)}$ does not change $m(A^{(n)} + B^{(n)})$, so we assume from now on that $x_n, y_n \in B^{(n)}$.

Claim 3. With a_n as above,

$$\liminf_{n\to\infty} a_n^{-1} m(B^{(n)}) = c > 0. \tag{3.2}$$

Proof of Claim 3. For each n, let $c_n \in \mathbb{N}$ such that $\frac{1}{3} \leq c_n a_n \leq \frac{2}{3}$. Assuming $c = 0$ in (3.2), then also $\liminf_{n\to\infty} c_n m(B^{(n)}) = 0$. Passing to a subsequence and renumbering we assume $\lim_{n\to\infty} c_n m(B^{(n)}) = 0$. If z is a limit point of the sequence $c_n(x_n - y_n)$, then $\chi\big(c_n(y_n - x_n)\big) = c_n a_n$ and $\chi(z) \in (\frac{1}{3}, \frac{2}{3})$, so that $z \neq 0$. Combining the fact that $A^{(n)} + B^{(n)} \sim A+B$ with the continuity of $z \mapsto m\big(E \cap (E-z)\big)$ and Lemma 2.21, we have

$$\begin{aligned} m\big((A+B)\cup(A+B+z)\big) & \\ &\leq \lim_{n\to\infty} m\big((A^{(n)}+B^{(n)}) \cup (A^{(n)}+B^{(n)}+c_n(x_n-y_n))\big) \\ &\leq \lim_{n\to\infty} m(A^{(n)}+B^{(n)}) + c_n m(B^{(n)}) \\ &= m(A+B). \end{aligned} \tag{3.3}$$

Inequality (3.3) implies $A+B+z \sim A+B$, contradicting the assumption that $H(A+B) = \{0\}$. $\square$

Claim 4. G is isomorphic to $F \times \mathbb{T}$, where F is a finite abelian group.

Proof of Claim 4. It suffices to show that the kernel G' of χ is discrete, and therefore finite. Let m' be Haar measure on G', so that

$$\int f\, dm = \int_G \int_{G'} f(z+w)\, dm'(w)\, dm(z)$$

for all bounded measurable $f : G \to \mathbb{R}$. Suppose, to get a contradiction, that G' is not discrete. Then there is a neighborhood U of $0 \in G$ with $m'\big((U+z)\cap G'\big) < c/2$ for all $z \in G$, where c is from (3.2). By Claim 1 there is an N such that

$$B^{(n)} - B^{(n)} \subset U \text{ for all } n > N.$$

Now

$$m(B^{(n)}) = \int m'\big(G' \cap (B^{(n)} - z)\big)\, dm(z).$$

The integrand is positive only if $z \in B^{(n)} - G' = B^{(n)} + G'$, so the integral can be estimated as

$$\begin{aligned} m(B^{(n)}) &= \int_{B^{(n)}+G'} m'\big(G' \cap (B^{(n)} - z)\big)\, dm(z) \\ &\leq m(B^{(n)} + G')m'(U) \\ &\leq a_n \cdot c/2, \end{aligned}$$

where the estimate $m(B^{(n)} + G') \leq a_n$ is a consequence of the fact that $\chi(B^{(n)})$ is contained in an interval of length a_n. Now the inequality $m(B^{(n)}) \leq a_n \cdot c/2$ for all $n > N$ contradicts Claim 3. □

We now complete the proof of Proposition 3.2 under the assumption that G is not totally disconnected and conclusion (i) holds in Lemma 3.1. By Claim 4, G is isomorphic to $F \times \mathbb{T}$ for some finite group F. Let G_0 be the connected component of the identity in G. By Claim 1, we can choose n sufficiently large that $B^{(n)}$ is contained in a coset of G_0. If $A^{(n)}$ is contained in a coset of G_0, then A and B are each contained in a coset of G_0. Then Theorem 1.2 implies the existence of a homomorphism $\chi : G_0 \to \mathbb{T}$ and intervals $I, J \subset \mathbb{T}$ such that $A \subset a + \chi^{-1}(I)$, $B \subset b + \chi^{-1}(J)$, for some $a, b \in G$, and $m(A) = m(\chi^{-1}(I)), m(B) = m(\chi^{-1}(J))$. Then we conclude (K) in Theorem 1.4. If $A^{(n)}$ is not contained in a coset of G_0, Lemma 2.14 then yields a quasi-periodic decomposition of $A^{(n)}$ with respect to G_0, and Lemmas 2.15 and 2.16 now imply that $A^{(n)} + B^{(n)}, A^{(n)}$, and $B^{(n)}$ are quasi-periodic with quasi-period G_0. Since $A^{(n)} + B^{(n)} \sim A + B$, we apply Part (ii) of Lemma 2.16 and conclude that (A, B) satisfies conclusion (QP) of Theorem 1.4. □

3.2. **Proof of Theorem 1.4.** To prove Theorem 1.4, assume (A, B) is a sur-critical pair for a compact abelian group G. If (A, B) is extendible, we have conclusion (E). If (A, B) is nonextendible and reducible, Corollary 2.5 implies (A, B) satisfies conclusion (QP). If (A, B) is irreducible and nonextendible, Lemma 2.11 allows us to assume that $A + B$ is measurable and $H(A + B) = \{0\}$. In this case, Proposition 3.2 implies the conclusion of Theorem 1.4. □

4. Reduction from locally compact to compact

The structure theorem for locally compact abelian groups allows one to identify a sur-critical pair in an arbitrary locally compact abelian group G with a sur-critical pair in a compact quotient of G. We state a convenient version of the structure theorem for reference.

Theorem 4.1. *If G is a locally compact abelian group, there is an open subgroup G_0 of G isomorphic to $K \times \mathbb{R}^n$, where K is a compact abelian group and $n \in \mathbb{N} \cup \{0\}$.* □

Lemma 4.2. *If G is a locally compact abelian group and $A, B \subset G$ are such that $m(A) > 0$, $m(B) > 0$ and $m(A + B) < \infty$, then A and B have compact closures.*

Proof. Assume, to get a contradiction, that the closure of A is not compact. It follows from Theorem 4.1 that there is a neighborhood U of $0 \in G$ and an infinite set $A' \subset A$ such that the sets $a' + U, a' \in A'$ are mutually disjoint. Now there is a $b \in B$ so that $m\big(B \cap (b + U)\big) > 0$, and $A + B$ contains all the mutually disjoint sets $a' + \big(B \cap (b + U)\big)$, where $a' \in A'$. Then $m(A + B) = \infty$. $\square$

Lemma 4.3. *Suppose $A, B \subset G$ have compact closures. Then there is a discrete cocompact subgroup $\Lambda \leqslant G$ such that $x + \Lambda = y + \Lambda$ implies $x = y$ for all $x, y \in A \cup B \cup (A + B)$, i.e. the quotient map $G \to G/\Lambda$ is one-to-one on $A \cup B \cup (A + B)$.*

Proof. By Theorem 4.1, G has a compact open subgroup G_0 isomorphic to $\mathbb{R}^n \times K$, where K is a compact abelian group, and the quotient G/G_0 is discrete. Since A and B have compact closures in G, the subgroup generated by $(A+G_0) \cup (B+G_0) \cup (A + B + G_0)$ in G/G_0 is finitely generated, with generators $e_1 + G_0, \dots, e_d + G_0$. Thus we may assume G/G_0 is finitely generated. Let $n_1, \dots, n_d$ be sufficiently large integers that the quotient map $G \to G/\langle n_1 e_1, \dots, n_d e_d\rangle$ is one-to-one on $A \cup B \cup (A + B)$. Now we may assume G/G_0 is finite, with generators $f_1, \dots, f_d$. Now choose a lattice $\Lambda \leqslant G_0$ so that $\Lambda \cap (A \cup B \cup (A + B)) \subset \{0_G\}$. Then the quotient map $G \to G/\Lambda$ is one-to-one on $A \cup B \cup (A + B)$, and the quotient is compact. $\square$

Corollary 4.4. *If G is a locally compact abelian group, and (A, B) is a sur-critical pair for G, then there is a compact group K, sur-critical pairs (C, D) for K with $m_K(C) = m_G(A)$, $m_K(D) = m_G(B)$, and a homomorphism $\psi : G \to K$ such that $\psi(A) = C$,$\psi(B) = D$, and ψ is one-to-one on $C \cup D \cup (C + D)$.*

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Department of Mathematics, The Ohio State University, 231 West 18th Ave., Columbus, OH 43210

E-mail address: griesmer.9@osu.edu